Chapter 4

Pure Bending

Pure Bending

This part is devoted to the study of pure bending and of the stresses and deformations it causes. In the picture below, the athlete holds the barbell with his hands placed at equal distances from the weights.

This results in **pure bending** in the center portion of the bar

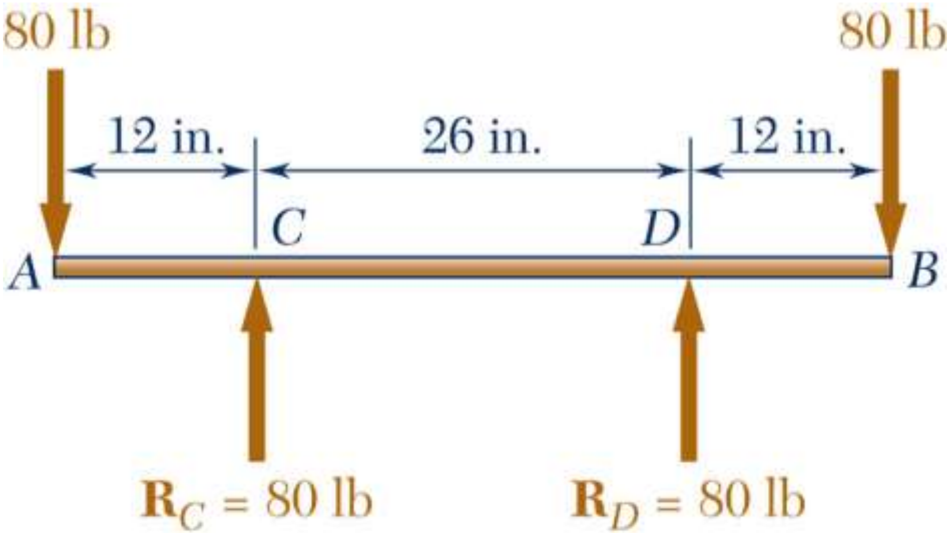
The **normal stresses** and the **curvature** resulting from **pure bending** will be determined in this part of the course.

We will account for stress concentrations where an abrupt change of geometry occurs in a flat bar

Pure Bending

Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

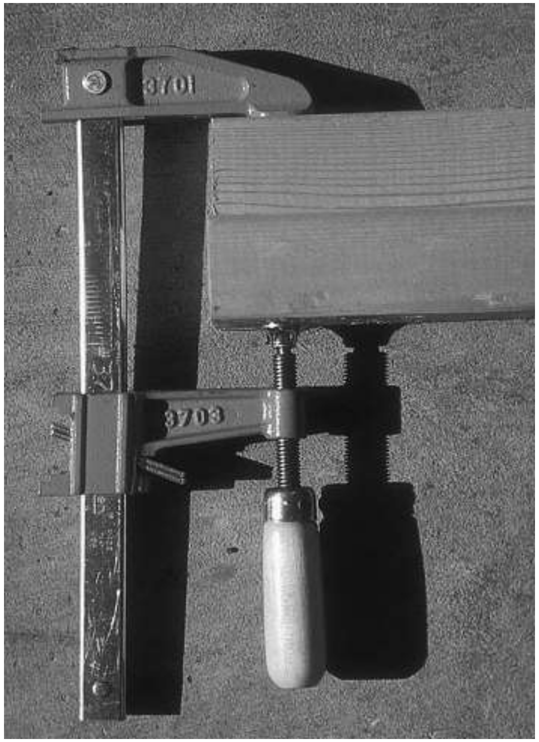
In the middle portion CD of the bar, the weight and reactions can be replaced by two equal and opposite **moment couples**



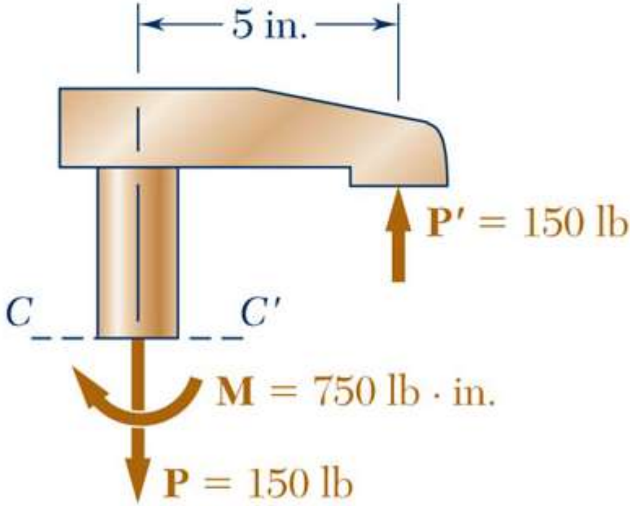
Bar Clamp

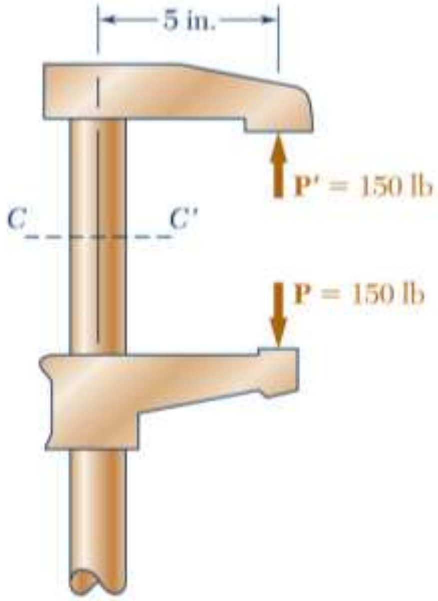
The bar clamp exerts forces on two pieces of lumber as they are being glued together. Equal and opposite forces are exerted by the lumber on the clamp. The straight portion of the clamp is subjected to an **eccentric loading**.

Eccentric Loading: Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple



Axial load
+
couple



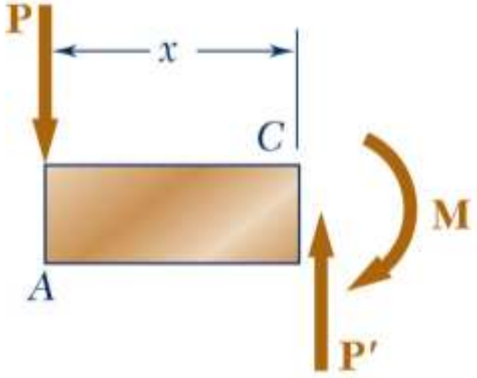


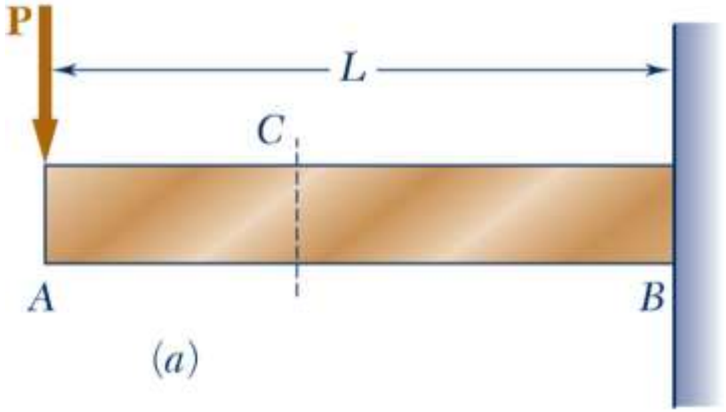
Cantilever Beam

The cantilever beam supports a **transverse load** P at its free end.

Transverse Loading: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple

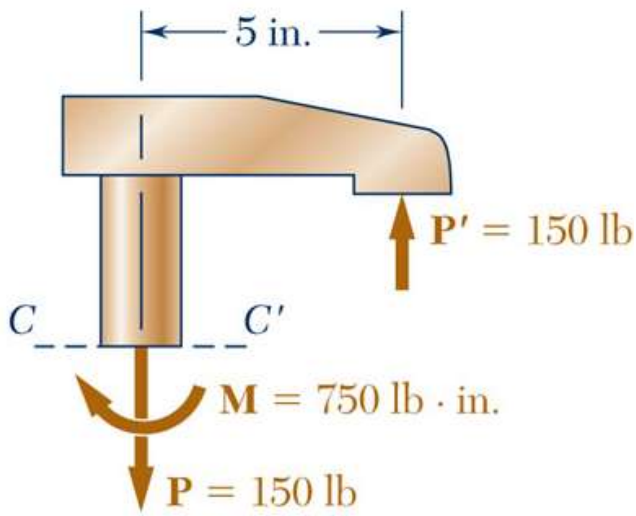
Shear force
+
couple

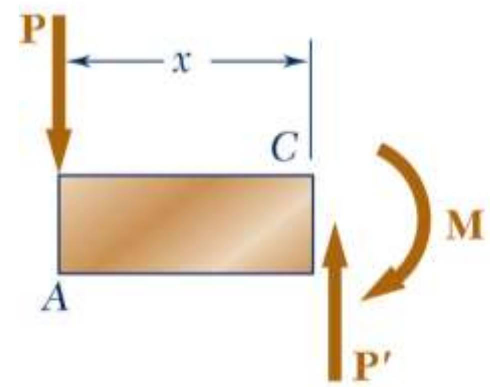




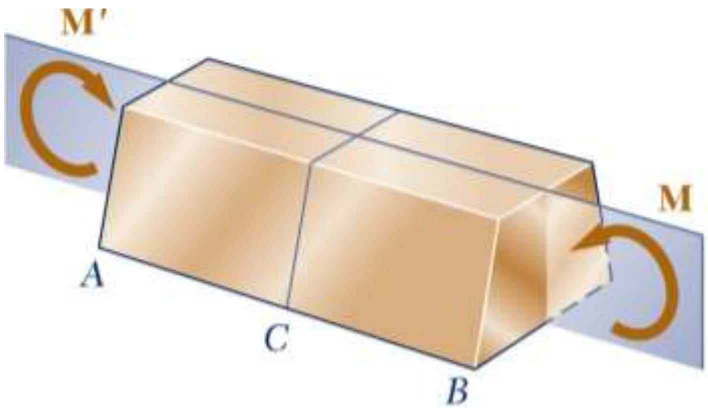
Principle of Superposition

Principle of Superposition: The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.



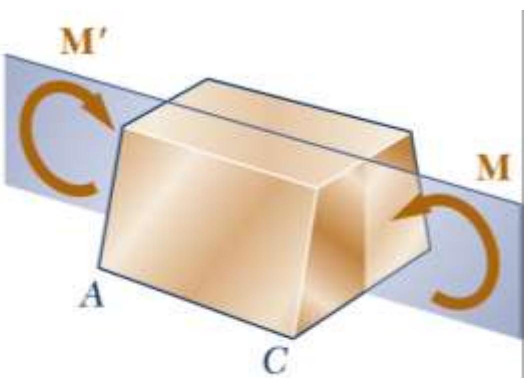


Symmetric Member in Pure Bending



Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section **bending moment**.

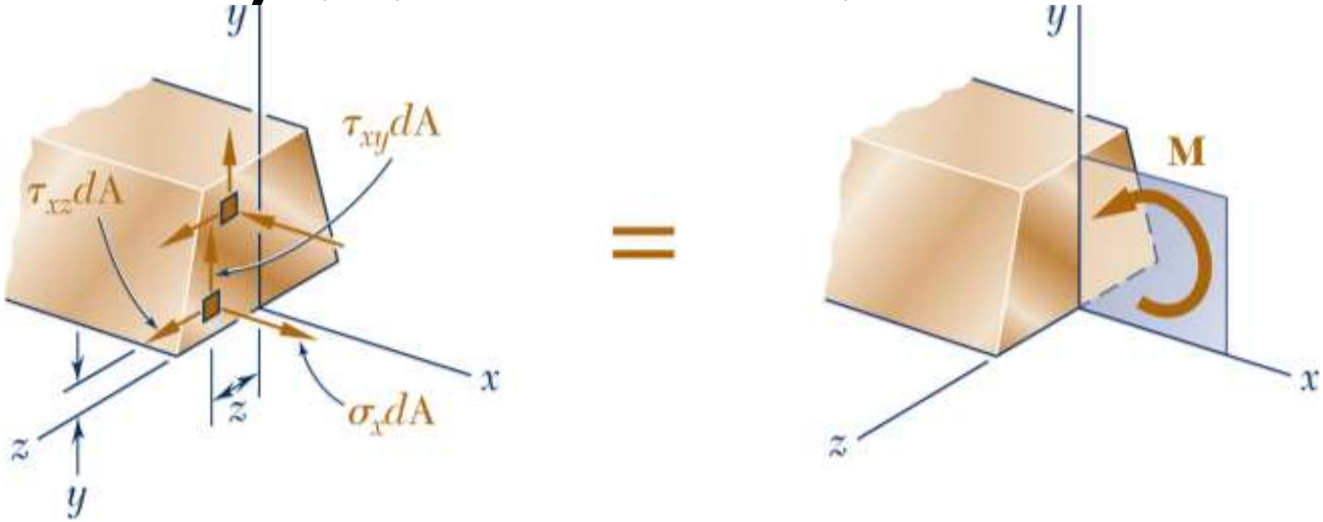
From statics, a couple M consists of two equal and opposite forces.



The sum of the components of the forces in any direction is zero.

The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.

Symmetric Member in Pure Bending



$$F_x = \int \sigma_x dA = 0$$

$$F_y = \int \tau_{xy} dA = 0$$

$$F_z = \int \tau_{xz} dA = 0$$

The sum of the components of the forces in any direction is equal to zero since no external forces are applied

$$M_x = \int \left(y\tau_{xz} - z\tau_{xy} \right) dA = 0$$

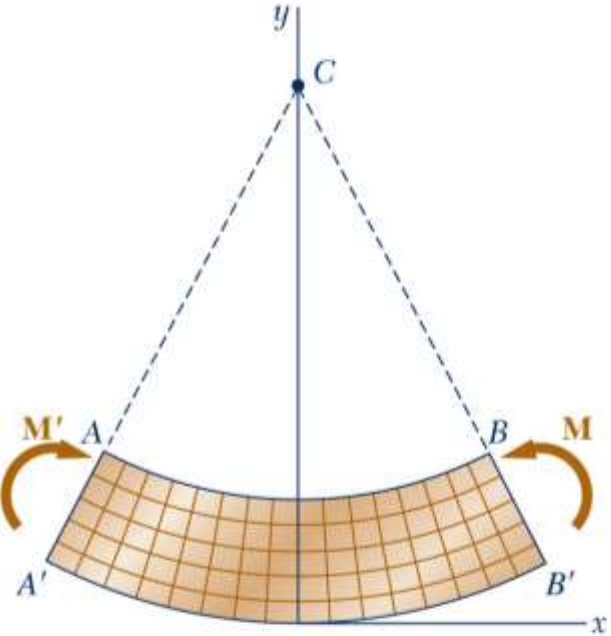
$$M_y = \int z\sigma_x dA = 0$$

$$M_z = \int \left(-y\sigma_x \right) dA = M$$

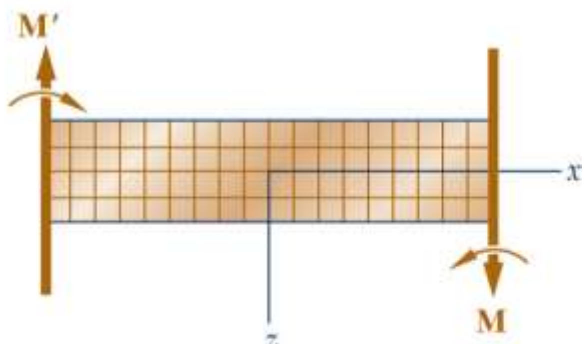
The sum of the moments of the elementary forces is equal to the external moments applied

The stress distribution in a given cross section cannot be determined from statics alone. It is **statically indeterminate** and may be obtained only by analyzing the deformations produced in the member.

Bending Deformations



(a) Longitudinal, vertical section (plane of symmetry)



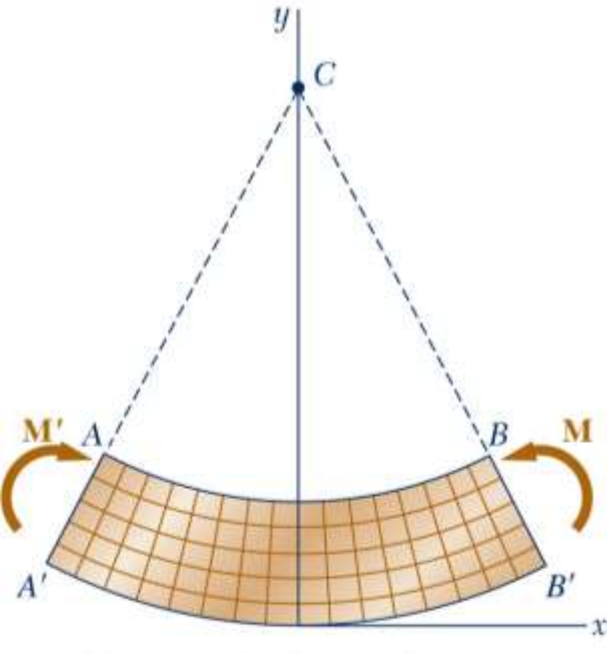
(b) Longitudinal, horizontal section

Beam with a plane of symmetry in **pure bending**:

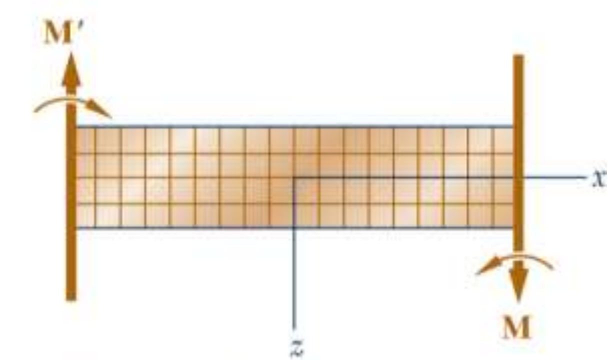
- Member remains symmetric
- Bends uniformly to form a circular arc
- Cross-sectional plane passes through arc center and remains planar
- All the faces keep 90º angles after deformation:

$$\gamma_{xy} = 0 \qquad \gamma_{xz} = 0 \quad \Longrightarrow \quad \tau_{xy} = 0 \qquad \tau_{xz} = 0$$

Bending Deformations



(a) Longitudinal, vertical section (plane of symmetry)



(b) Longitudinal, horizontal section

Also $\sigma_y = \sigma_z = \tau_{yz} = 0$

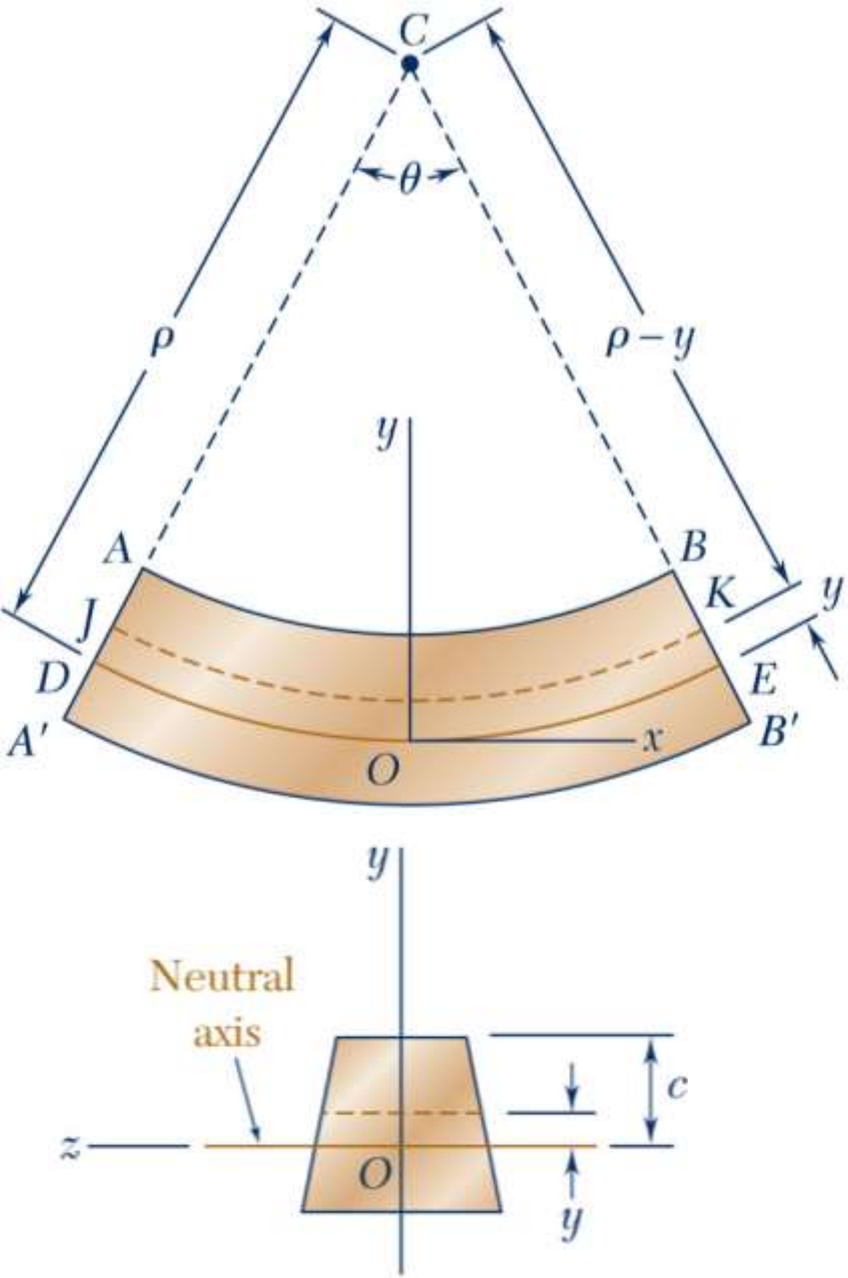
The length of top surface decreases and the length of bottom face increases. Hence, a **neutral surface** must exist that is parallel to the upper and lower surfaces and for which the length does not change.

In this scenario: $\mathbf{M_z > 0}$

The surfaces **above** the neutral surface are under **compression**, i.e. stresses and strains are **negative**.

The surfaces **below** the neutral surface are under **tension**, i.e. stresses and strains are **positive**.

Strain Due to Bending



(b) Transverse section

Neutral axis: Arc DE is the intersection of the neutral surface and the plane of symmetry

The strain at a distance y from E:

Initial length $L = \rho\theta$

Final length $L' = (\rho - y)\theta$

Deformation $\delta = L' - L = (\rho - y)\theta - \rho\theta$

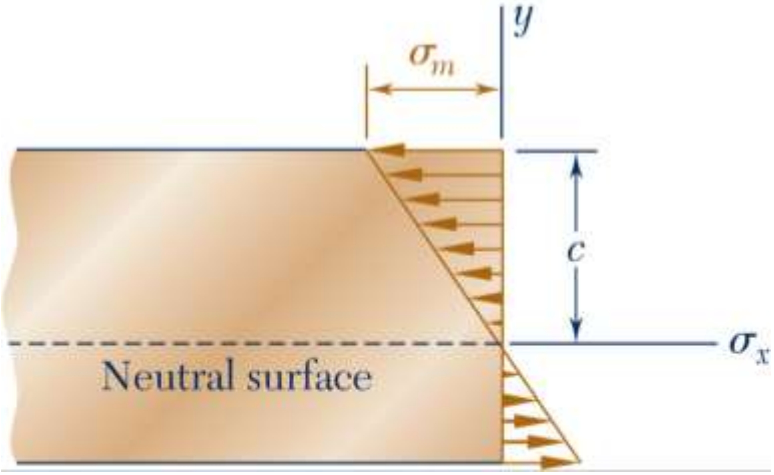
$$\delta = -y\theta$$

Strain $\varepsilon_x = \dfrac{\delta}{L} = -\dfrac{y\theta}{\rho\theta} = -\dfrac{y}{\rho}$

Max Strain $\varepsilon_m = \dfrac{c}{\rho}$

$$\Longrightarrow \quad \varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Stress Due to Bending



For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$\sigma_x = -\frac{y}{c}\sigma_m$$

For equilibrium,

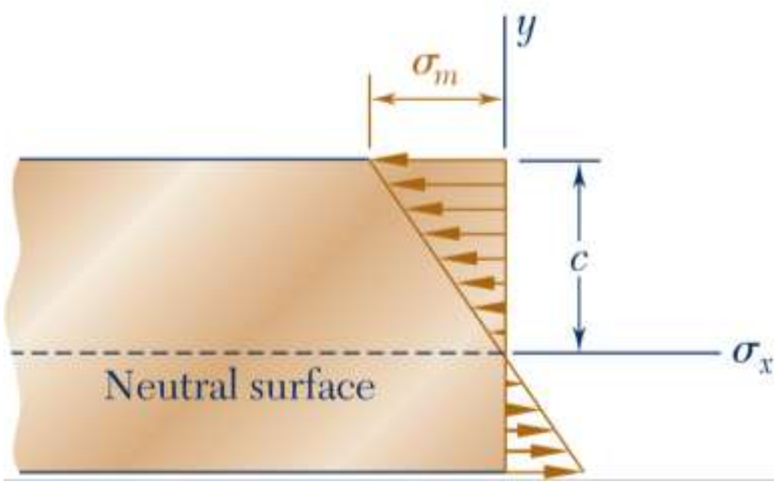
$$F_x = 0$$

$$\int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA = 0 \quad \Longrightarrow \quad \int y \, dA = A\bar{y} = 0$$

$$\bar{y} = 0$$

First moment with respect to neutral plane is zero.
Therefore, the **neutral surface** must pass through the **section centroid**.

Stress Due to Bending



For static equilibrium,

$$M = \int(-y\sigma_x \, dA) = \int(-y)\left(-\frac{y}{c}\sigma_m\right) dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

I is the **moment of inertia** or second moment of the cross section
S is the **elastic section modulus**

$$I = \int y^2 \, dA$$

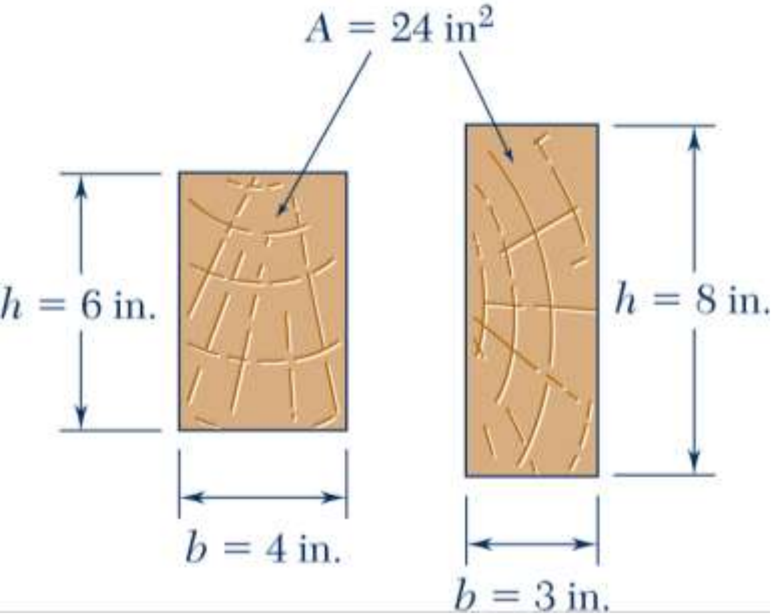
$$S = \frac{I}{c}$$

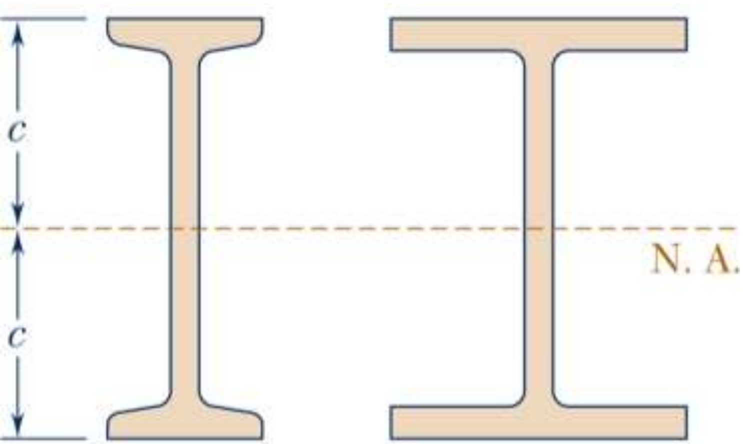
$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$$\sigma_x = -\frac{y}{c}\sigma_m$$

$$\Longrightarrow \quad \sigma_x = -\frac{My}{I}$$

Beam Section Properties





(*a*) S-beam (*b*) W-beam

The maximum normal stress due to bending,

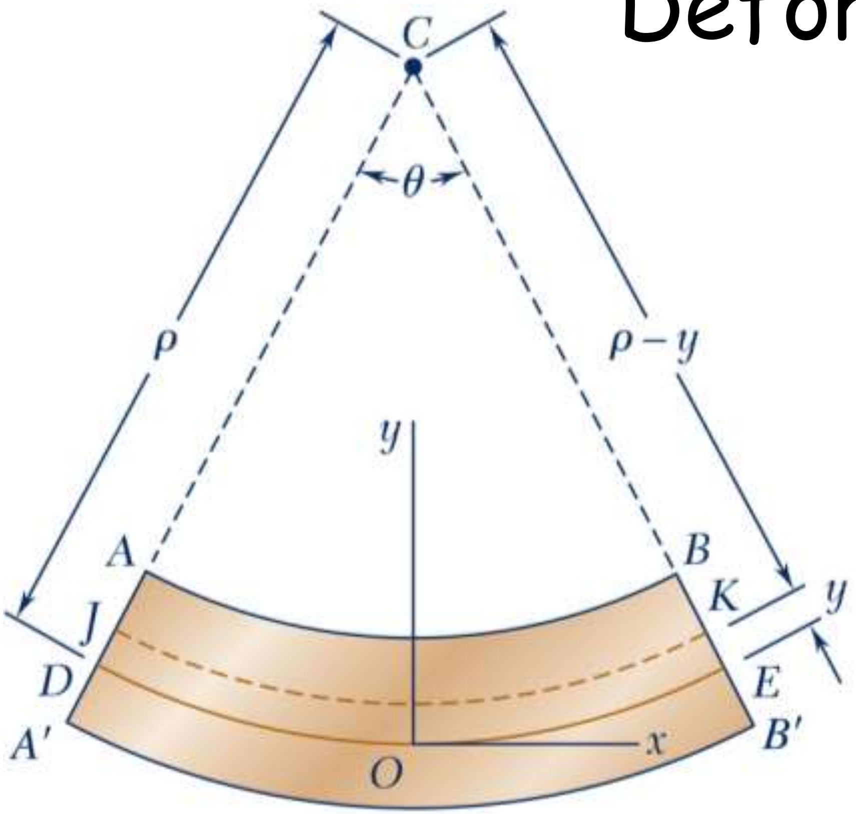
$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

A beam section with a **larger section modulus** will have a **lower maximum stress**. Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \tfrac{1}{6}bh^2 = \tfrac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending. Structural steel beams are designed to have a large section modulus

Deformation Due to Bending



The deformation of the member by the bending moment M is measured by the **curvature of the neutral surface**

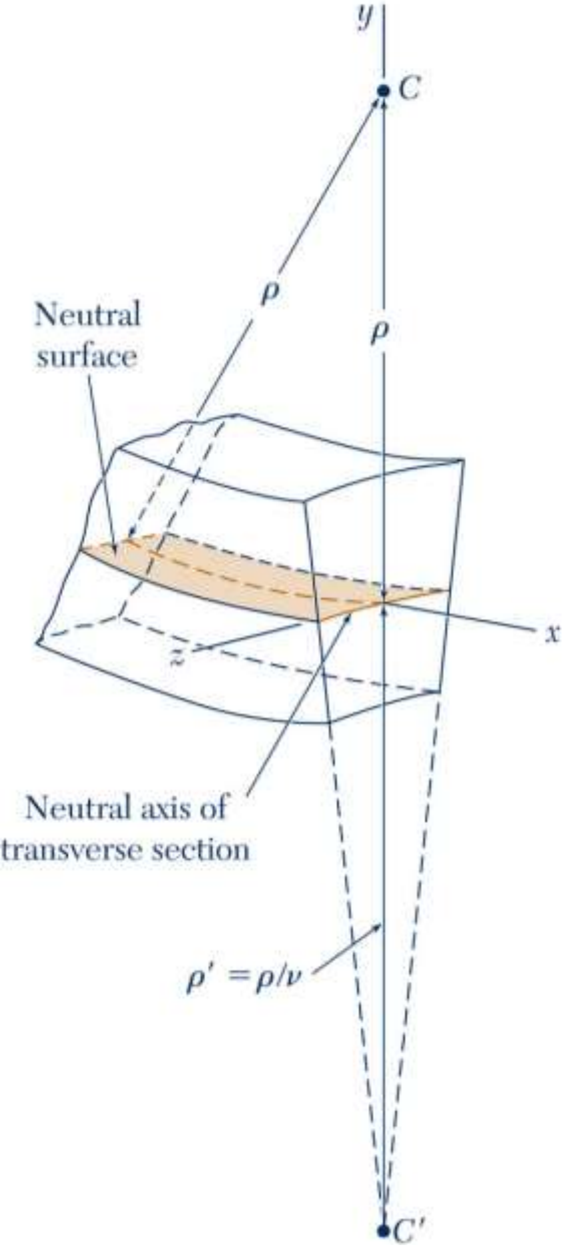
$$\frac{1}{\rho} = \frac{\varepsilon_m}{c}$$

$$\varepsilon_m = \frac{\sigma_m}{E}$$

$$\sigma_m = \frac{Mc}{I}$$

$$\Longrightarrow \quad \frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I} \quad \Longrightarrow \quad \boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

Deformations: Transverse Cross-Section



Deformation due to bending moment M is quantified by the **curvature of the neutral surface**

$$\frac{1}{\rho} = \frac{M}{EI}$$

Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

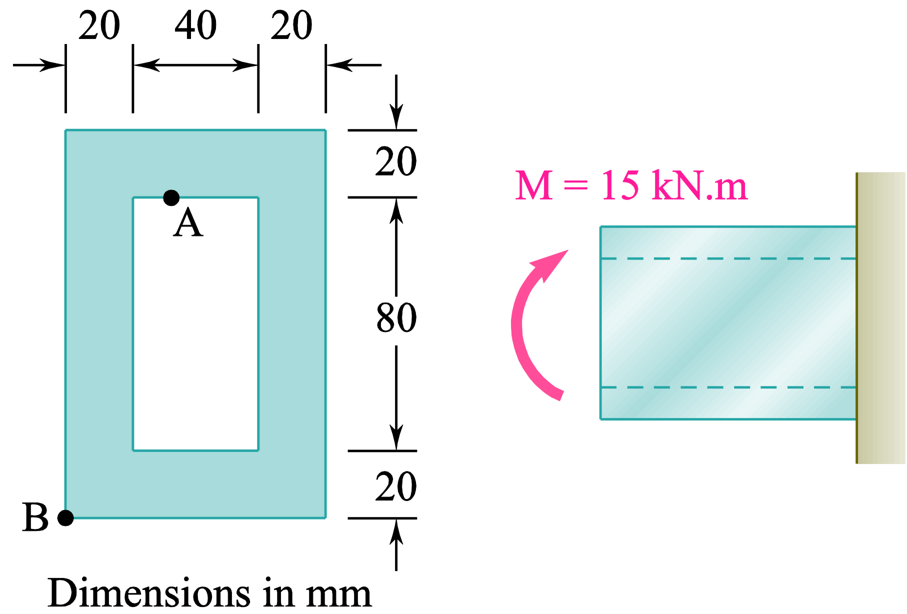
$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

Expansion above the neutral surface and contraction below it cause an in-plane curvature. The curvature of the transverse cross section is called **anticlastic curvature**

$$\frac{1}{\rho'} = \frac{\nu}{\rho}$$

Example 1

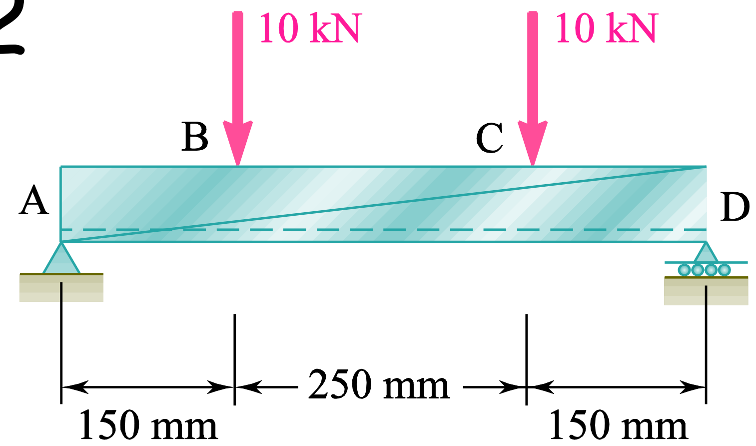
4-1: Knowing that the couple shown acts in a vertical plane, determine the stress at **(a)** point *A*, **(b)** point *B*.

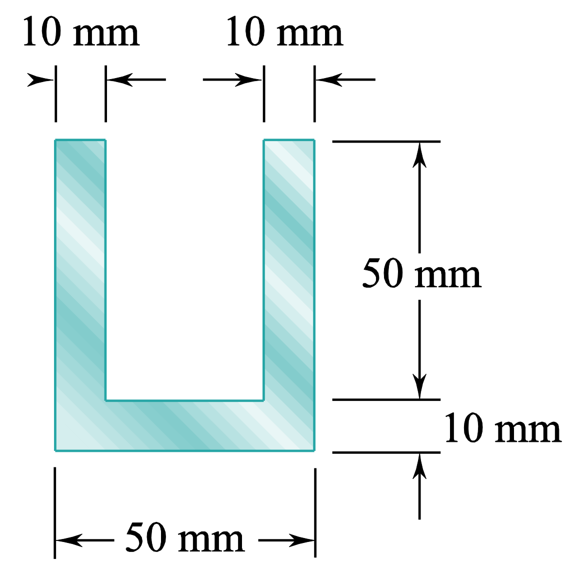


Example 1

Example 2

4-11: Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.

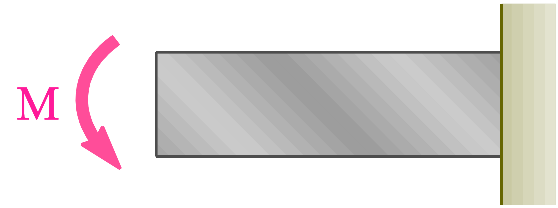


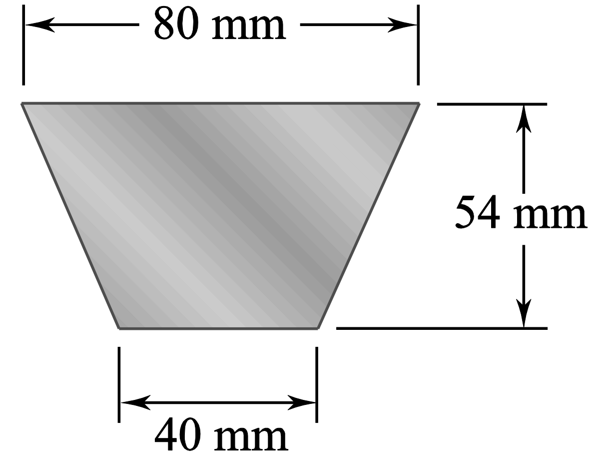


Example 2

Example 3

4-19: Knowing that for the extruded beam shown the allowable stress is 120 MPa in tension and 150 MPa in compression, determine the largest couple ***M*** that can be applied.

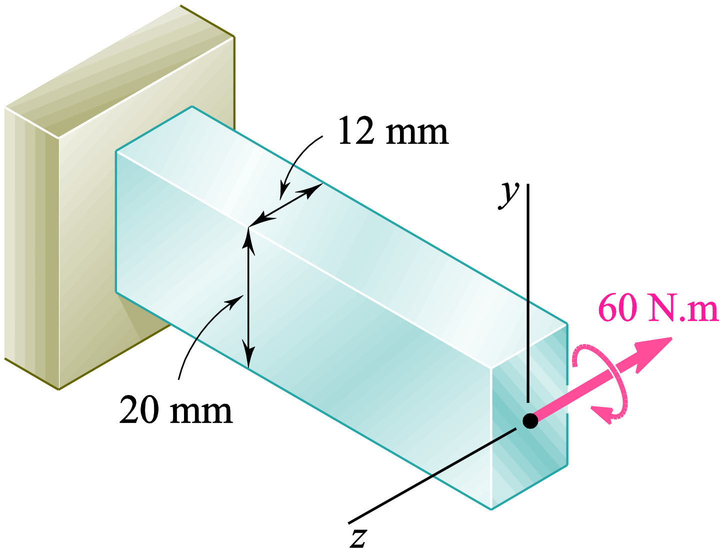




Example 3

Example 4

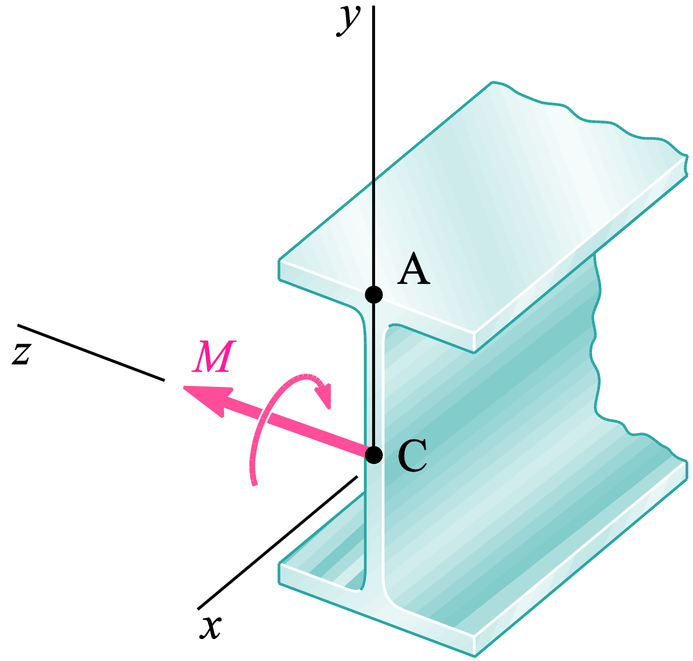
4-24: A 60 N.m couple is applied to the steel bar shown. **(a)** Assuming that the couple is applied about the z axis as shown, determine the maximum stress and the radius of curvature of the bar. **(b)** Solve part a, assuming that the couple is applied about the y axis. Use E = 200 GPa.



Example 4

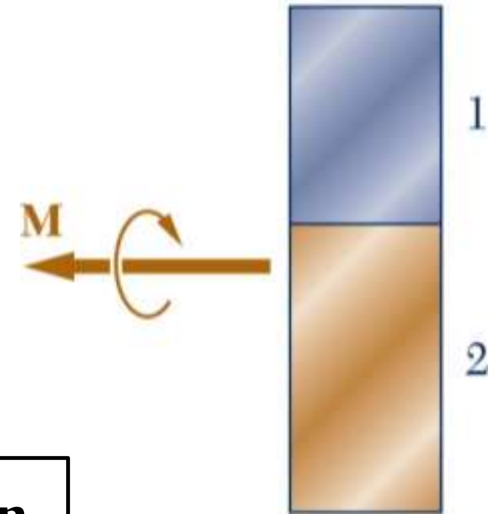
Example 5

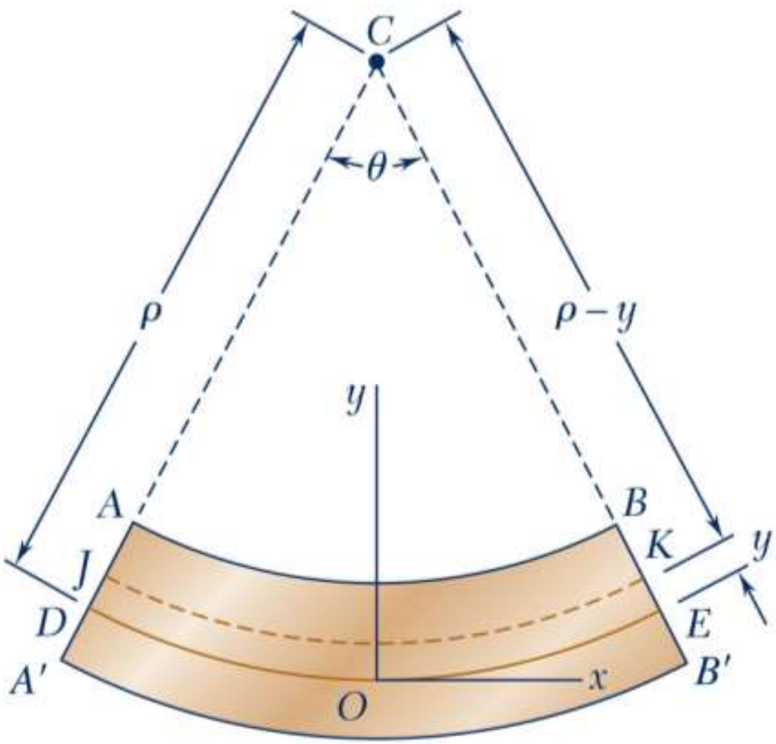
4-31: A W200 × 31.3 rolled-steel beam is subjected to a couple **M** of moment 45 kN.m. Knowing that E = 200 GPa and ν = 0.29, determine **(a)** the radius of curvature ρ, **(b)** the radius of curvature ρ' of a transverse cross section.



Members Made of Several Materials

Consider a composite beam formed from two materials with $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$. The normal strain still **varies linearly** with the distance y from the neutral axis. The stress in material 1 will be different than that in material 2.





Normal Strain

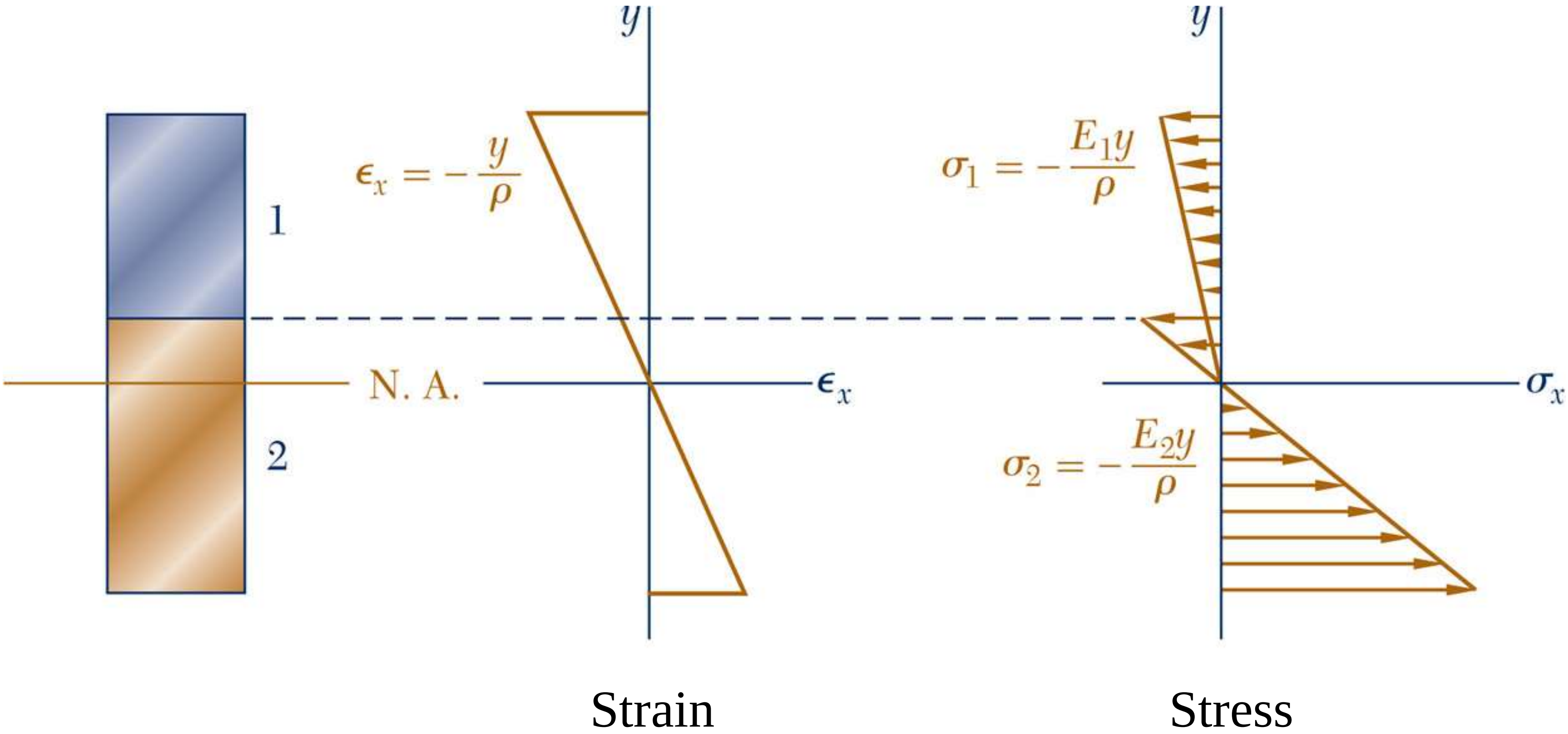
$$\varepsilon_x = -\frac{y}{\rho}$$

Normal Stress

$$\sigma_1 = E_1 \varepsilon_x = -\frac{E_1 y}{\rho}$$

$$\sigma_2 = E_2 \varepsilon_x = -\frac{E_2 y}{\rho}$$

Members Made of Several Materials



The **neutral axis does not pass** through the **section centroid** of the composite section.

Members Made of Several Materials

Static Equilibrium: The total normal force **F** resulting from the stress distribution normal to the cross section should equal to zero:

$$F = F_1 + F_2 = 0$$

The total force $\mathbf{F_1}$ exerted on the cross sectional area $\mathbf{A_1}$ of **material 1** is:

$$F_1 = \int_{A_1} \sigma_1 dA = \int_{A_1} -\frac{E_1 y}{\rho} dA$$

The total force $\mathbf{F_2}$ exerted on the cross sectional area $\mathbf{A_2}$ of **material 2** is:

$$F_2 = \int_{A_2} \sigma_2 dA = \int_{A_2} -\frac{E_2 y}{\rho} dA$$

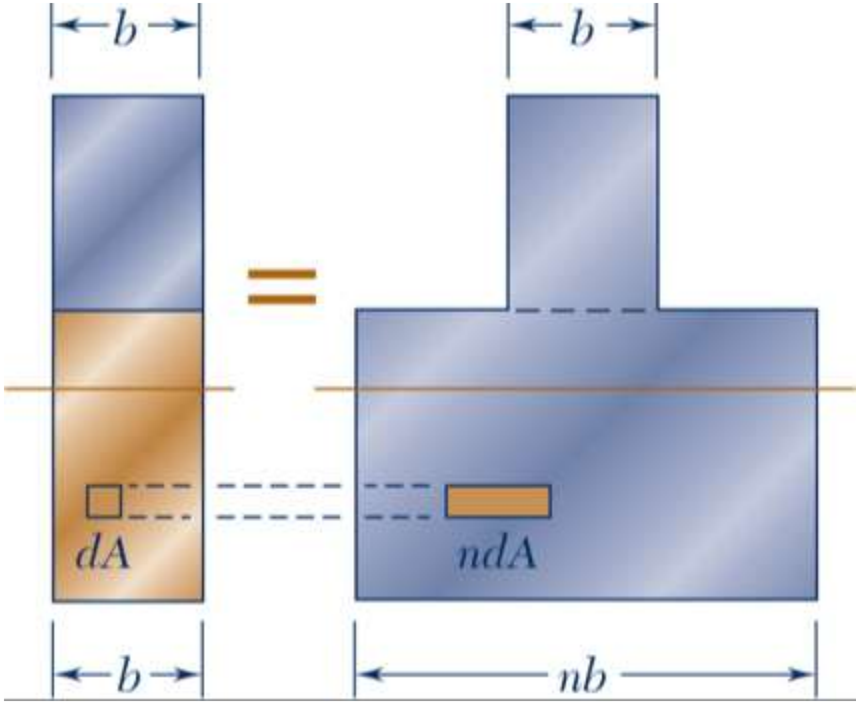
Members Made of Several Materials

The location of the neutral surface is found from the static equilibrium equation:

$$F = \int_{A_1} -\frac{E_1 y}{\rho} dA + \int_{A_2} -\frac{E_2 y}{\rho} dA = 0$$

This equation can be easily solved as follows: First let

$$n = \frac{E_2}{E_1}$$



$$F = \int_{A_1} -\frac{E_1 y}{\rho} dA + \int_{A_2} -\frac{nE_1 y}{\rho} dA = 0$$

$$F = \int_{A_1} -\frac{E_1 y}{\rho} dA + \int_{A_2} -\frac{E_1 y}{\rho} (ndA) = 0$$

If you now multiply the dimension of $\boldsymbol{A_2}$ parallel to the neutral surface by n, the force equation can be rewritten as:

$$F = \int_{A_1} -\frac{E_1 y}{\rho} dA + \int_{A_2'} -\frac{E_1 y}{\rho} dA = 0$$

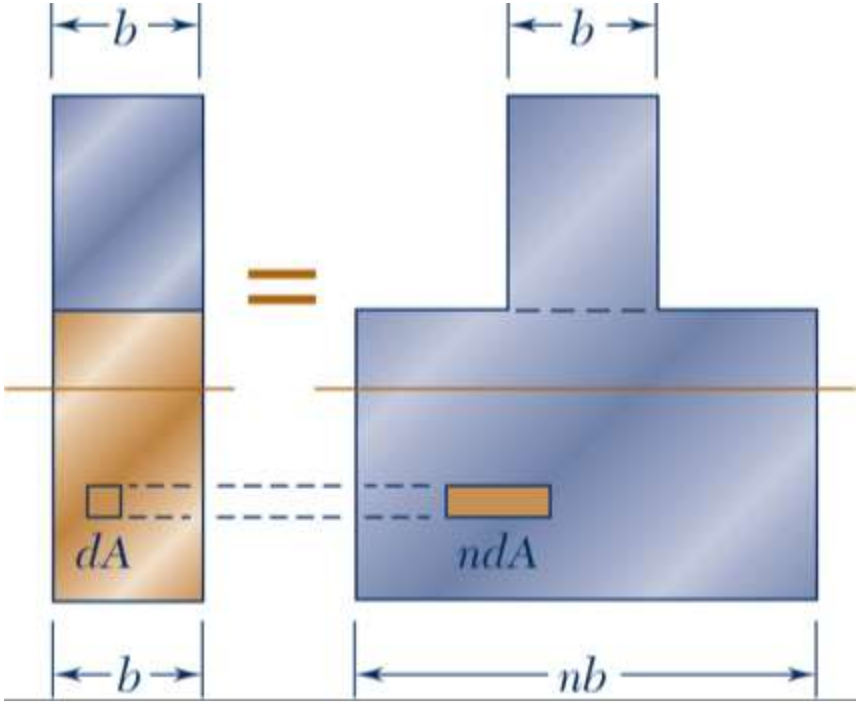
Members Made of Several Materials

Then the equation is simplified to:

$$F = \int_{A'} -\frac{E_1 y}{\rho}\, dA = 0$$

The new converted cross section is:

$$A' = A_1 + A_2'$$

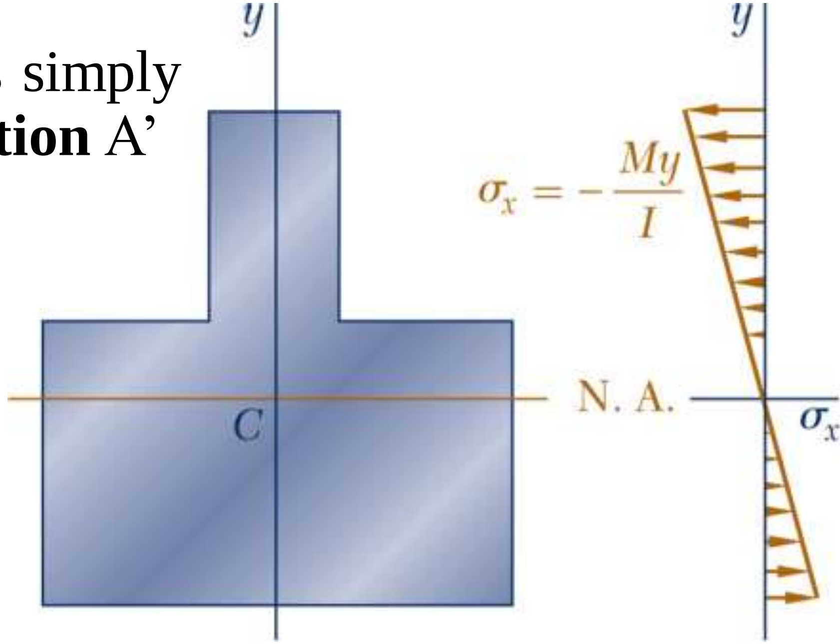


The integral of ***F*** depends now on $\mathbf{E_1}$ only. Physically, it defines the summation of the normal forces resulting from the application of the normal stresses on ***A'*** made of **material 1**. As if you converted the cross-section $\boldsymbol{A_2}$ made of **material 2** to another cross-section $\boldsymbol{A_2'}$ made of **material 1**.

Since it is essential for the distance y of each element from the neutral axis to **remain the same**, the dimension parallel the neutral axis should change to n×b.

Members Made of Several Materials

The neutral surface of the cross section A is simply the **centroid** of the **new converted cross section** A’



The stress σ_x at any point of the transformed member is obtained as

$$\sigma_x = -\frac{My}{I}$$

The stress $\boldsymbol{\sigma_1}$ at any point in **material 1** of the original member is obtained as follows,

$$\sigma_1 = \sigma_x$$

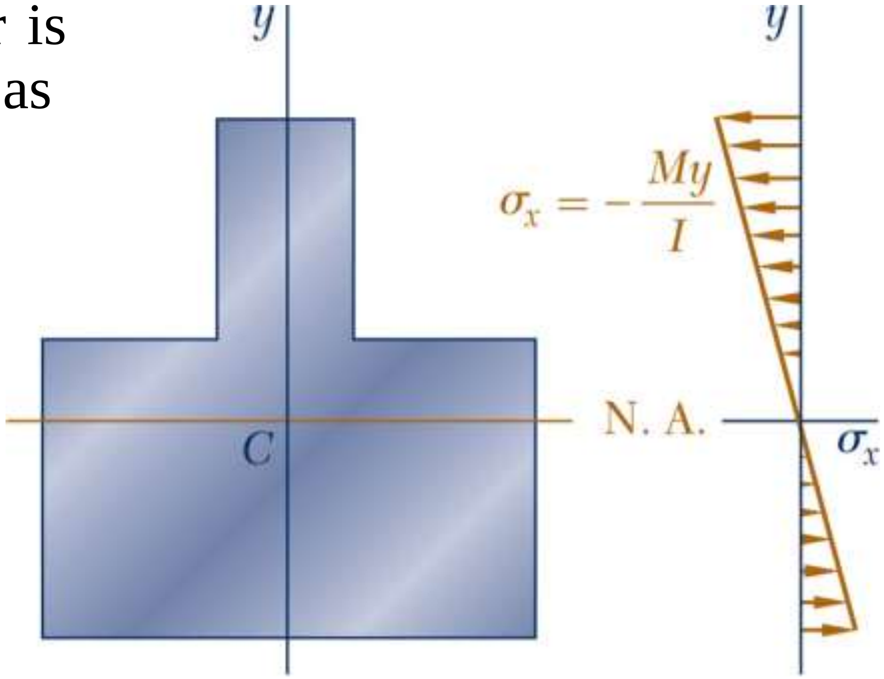
The stress $\boldsymbol{\sigma_2}$ at any point in **material 2** of the original member is calculated as:

$$\sigma_2 = n\sigma_x$$

Original	**Transformed**
Area	n×Area
Force	Force

Members Made of Several Materials

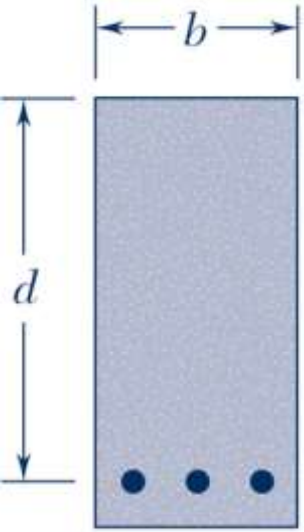
The deformation of a composite member is determined using the transformed section as



$$\frac{1}{\rho} = \frac{M}{E_1 I}$$

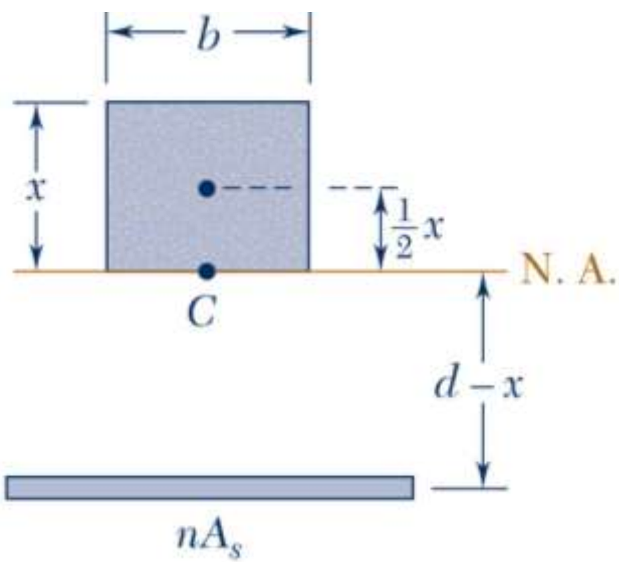
E_1 is the Young's modulus of **material 1**
I is the moment of inertia of the transformed section with respect to its neutral axis

Reinforced Concrete Beams



Concrete beams subjected to bending moments are reinforced by steel rods. The steel rods carry the entire tensile load below the neutral surface. The upper part of the concrete beam carries the compressive load only since concrete is weak in tension.

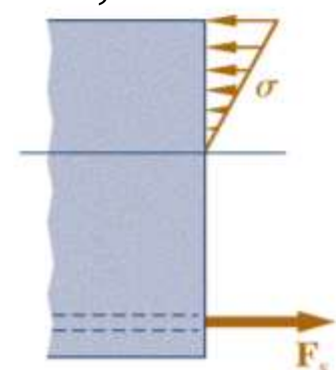
In the transformed section, the cross sectional area of the steel, A_s, is replaced by the equivalent area nA_s where $n = E_s/E_c$.



To determine the location of the neutral axis,

$$(bx)\frac{x}{2} - nA_s(d-x) = 0$$

$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0$$



The normal stress in the concrete and steel

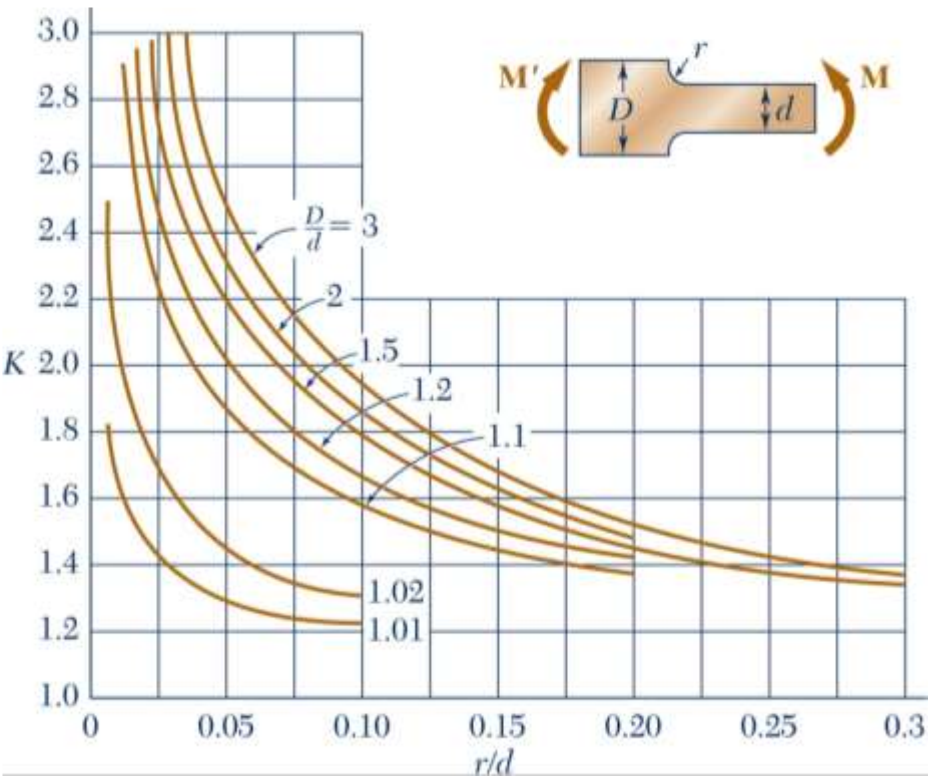
$$\sigma_x = -\frac{My}{I} \qquad \sigma_s = n\sigma_x$$

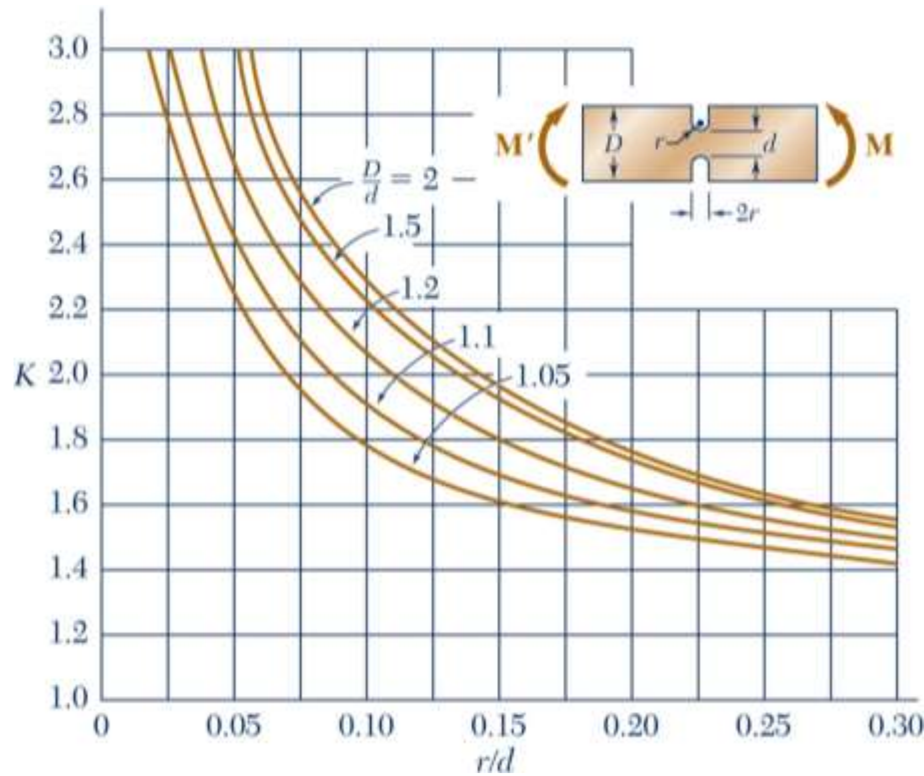
Stress Concentrations

Stress concentrations may occur in the vicinity of :

- points where the loads are applied
- abrupt changes in cross section

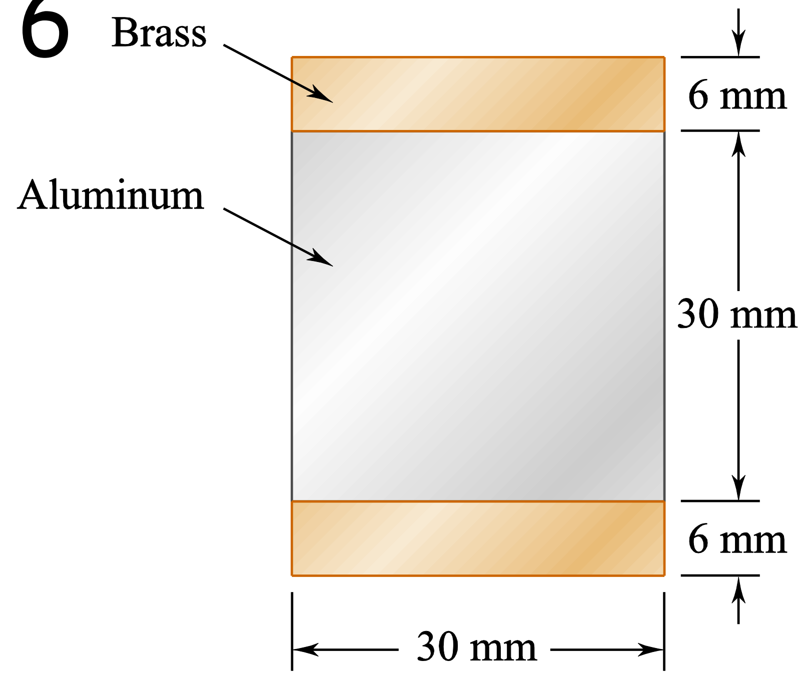
$$\sigma_m = K\frac{Mc}{I}$$





Example 6

4-33: A bar having the cross section shown has been formed by securely bonding brass and aluminum stock. Using the data given below, determine the largest permissible bending moment when the composite bar is bent about a horizontal axis

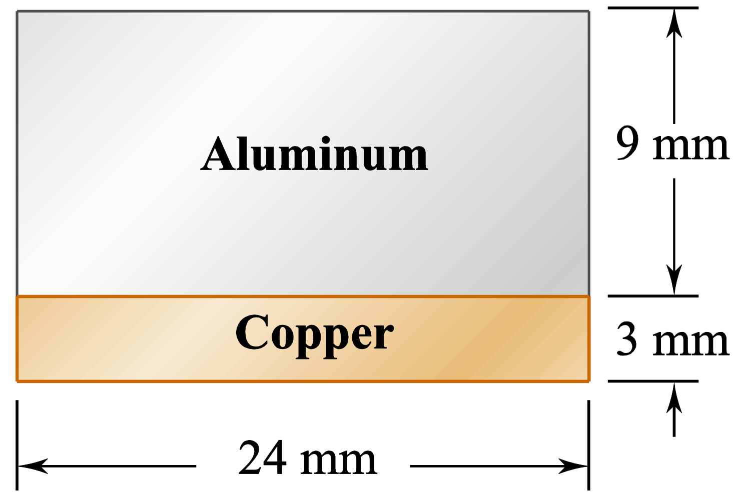


	Aluminum	Brass
Modulus of elasticity	70 GPa	105 GPa
Allowable stress	100 MPa	160 MPa

Example 6

Example 7

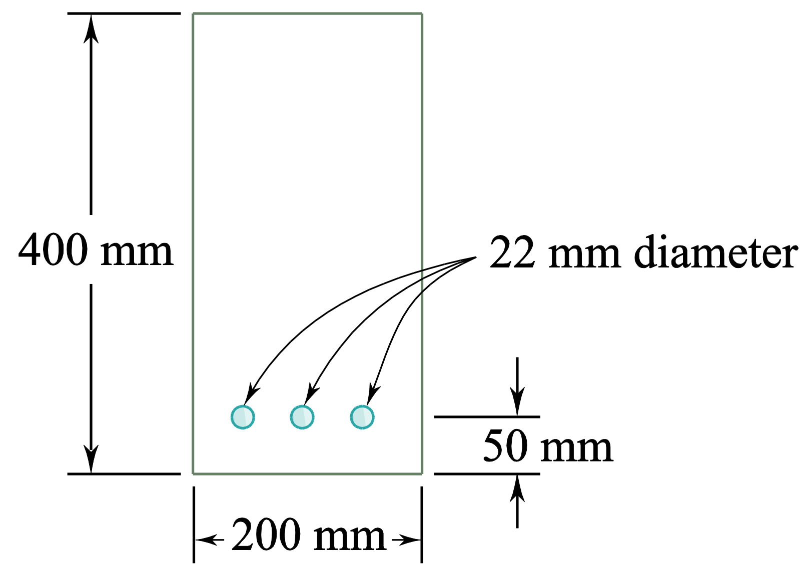
4-40: A copper strip (E_c = 105 GPa) and an aluminum strip (E_a = 75 GPa) are bonded together to form the composite beam shown. Knowing that the beam is bent about a horizontal axis by a couple of moment M = 35 N.m, determine the maximum stress in **(a)** the aluminum strip, **(b)** the copper strip.



Example 7

Example 8

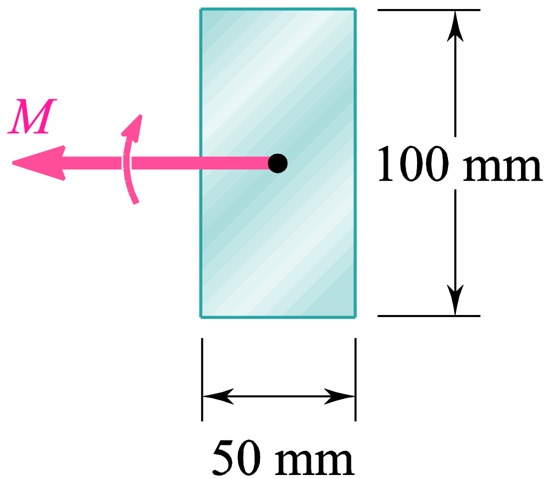
4-52: A concrete beam is reinforced by three steel rods placed as shown. The modulus of elasticity is 20 GPa for the concrete and 200 GPa for the steel. Using an allowable stress of 9 MPa for the concrete and 140 MPa for the steel, determine the largest allowable positive bending moment in the beam.

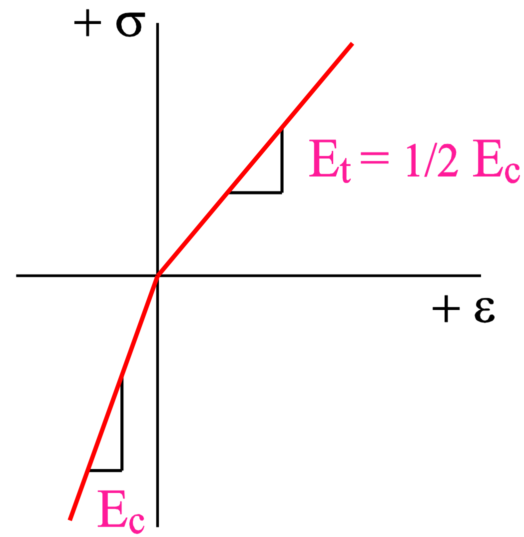


Example 8

Example 9

4-59: The rectangular beam shown is made of a plastic for which the value of the modulus of elasticity in tension is one-half of its value in compression. For a bending moment M = 600 N.m, determine the maximum **(a)** tensile stress, **(b)** compressive stress.

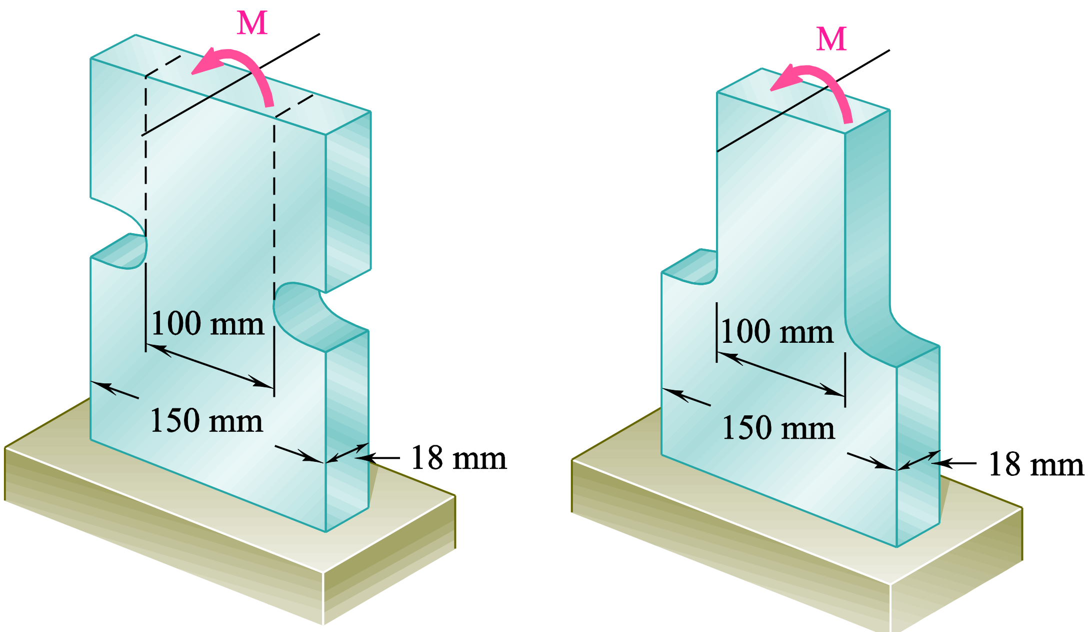




Example 9

Example 10

4-65: A couple of moment M = 2 kN.m is to be applied to the end of a steel bar. Determine the maximum stress in the bar **(a)** if the bar is designed with grooves having semicircular portions of radius r = 10 mm, as shown in Fig. a, **(b)** if the bar is redesigned by removing the material to the left and right of the dashed lines as shown in Fig. b.



(a) (b)

Example 10